PROJECTED WRITTEN NOTES FROM THE M325K LECTURE ON THURSDAY, JANUARY 18, 2024, on Logic, Statement forms, TRUTH TABLES, COMPOUND STATEMENTS, EQUIVALENT STATEMENT FORMS, DEMORGAN'S LAWS, AND CONDITIONAL STATEMENTS CLASS #2

In the handout "Defining Variables in a Proof", in your solution to the problem. you have 3 tasks to accomplish:

1. List the variables which are used without being adequately defined.
2. Explain, in each instance, why the variable fails to be defined
3. Add ~~ins~~ statements that make it so that each variable is adequately defined when it is used.

Note: Some proof segments have incorrect logic but DO NOT Try to Fix the Logic — Only make it so that all the variables are adequately defined when used.

Logic - Rules of Deduction by which we derive the Truth of a statement from an argument (a.k.a. "proof") based on true premises.

A statement is a declarative sentence which can meaningfully be considered as True or False, (T) (F) but not both.

Statements	Non-Statements
Our class meets in the PMA building.	Go to the store!
Today is Sunday.	Running is better exercise than swimming.
$7 > 25$.	This statement is false.

Command

A matter of opinion

A Truth Value cannot meaningfully be assigned.

A compound statement is a statement with a structure derived from other statements, using connectives (e.g.: AND, OR, NOT, etc)

Statement Forms

EXAMPLES:

A conjunction.

$p \wedge q$ — $\underbrace{\text{Today is Thursday}}_{p}$ AND $\underbrace{\text{we must go to school}}_{q}$.

p AND q

$r \vee s$ — $\underbrace{\text{I'm telling the Truth}}_{r}$ OR $\underbrace{\text{my name is Napoleon}}_{s}$.

r OR s

A DISJUNCTION

$\sim h$ — My Name is Not Frank.

NOT h

The Negation of $\underbrace{\text{"My Name is Frank"}}_{h}$

$p \wedge (q \vee s)$ — Today is Thursday AND Either we must go to school or My Name is Napoleon

$(p \wedge q) \vee s$ — Either Today is Thursday and we must go to school OR My Name is Napoleon.

Aristotelean Logic is a two-valued logic.

A Truth Value T or F is assigned a statement according to whether it is True or False.

The Truth value of a compound statement depends on the Truth values of its component statements according to certain rules:

The Truth Table of a compound statement shows the truth value that the compound statement has in all the possible ways that Truth values can be assigned to the basic statements in the compound statement.

The <u>NEGATION</u> OF statement p is a statement $\sim p$ (called "NOT p") which in all cases has the opposite truth value as p has.

p	$\sim p$
T	F
F	T

<u>Some rules for assigning truth values:</u>

p	q	$p \wedge q$ ("p AND q")	$p \vee q$ ("p OR q")	$\sim p$
T	T	T	T	F
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Exercise:

Determine the truth table of the statement form: $(p \wedge \sim q) \vee s$

THE TRUTH TABLE FOR $(p \wedge \sim q) \vee s$

p	q	s	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \vee s$
T	T	T	F	F	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	T	F	F	F	F
F	F	T	T	F	T
F	F	F	T	F	F

EXAMPLE: Either this Monday is a holiday and we don't have to go to school or I will stay home anyway.

Question: UNDER WHAT CONDITIONS WILL IT be a lie to say this?

Use p = "Monday is a holiday."
q = "We have to go to school."
s = "I stay home."

Definition: Two statement forms (with the same variables) are *logically equivalent* ($\equiv$) if they have the same truth table (and so they have the same truth value in all possible cases).

EXAMPLE:

DE MORGAN'S LAWS:

(1) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

(2) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

Proof of (2):

p	q	$(p \vee q)$	$\sim(p \vee q)$	$(\sim p)$	$\wedge$	$(\sim q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	T
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Because these are the same, $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

is equivalent to

$$(\textit{file\_modification\_date} < \textit{date}1) \quad \text{or} \quad (\textit{date}2 < \textit{file\_modification\_date}).$$

Example 2.1.11 A Cautionary Example

According to De Morgan's laws, the negation of

$$p\text{: Jim is tall and Jim is thin}$$

is

$$\sim p\text{: Jim is not tall or Jim is not thin}$$

because the negation of an *and* statement is the *or* statement in which the two components are negated.

Unfortunately, a potentially confusing aspect of the English language can arise when you are taking negations of this kind. Note that statement p can be written more compactly as

$$p'\text{: Jim is tall and thin.}$$

When it is so written, another way to negate it is

$$\sim(p')\text{: Jim is not tall and thin.}$$

Caution! Although the laws of logic are extremely useful, they should be used as an *aid* to thinking, not as a mechanical substitute for it.

But in this form the negation looks like an *and* statement. Doesn't that violate De Morgan's laws?

Actually no violation occurs. The reason is that in formal logic the words *and* and *or* are allowed only between complete statements, not between sentence fragments.

One lesson to be learned from this example is that when you apply De Morgan's laws, you must have complete statements on either side of each *and* and on either side of each *or*. ■

Tautologies and Contradictions

It has been said that all of mathematics reduces to tautologies. Although this is formally true, most working mathematicians think of their subject as having substance as well as form. Nonetheless, an intuitive grasp of basic logical tautologies is part of the equipment of anyone who reasons with mathematics.

p.34

• Definition

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement.**

A **contradication** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradication is a **contradictory statement.**

According to this definition, the truth of a tautological statement and the falsity of a contradictory statement are due to the logical structure of the statements themselves and are independent of the meanings of the statements.

8

FROM the TEXTBOOK

④

Example 2.1.12 Tautologies and Contradictions

Show that the statement form $p \vee \sim p$ is a tautology and that the statement form $p \wedge \sim p$ is a contradiction.

Solution

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

all T's so $p \vee \sim p$ is a tautology

all F's so $p \wedge \sim p$ is a contradiction

Example 2.1.13 Logical Equivalence Involving Tautologies and Contradictions

If $\mathbf{t}$ is a tautology and $\mathbf{c}$ is a contradiction, show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$.

Solution

p	$\mathbf{t}$	$p \wedge \mathbf{t}$	p	$\mathbf{c}$	$p \wedge \mathbf{c}$
T	T	T	T	F	F
F	T	F	F	F	F

same truth values, so $p \wedge \mathbf{t} \equiv p$

same truth values, so $p \wedge \mathbf{c} \equiv \mathbf{c}$

Summary of Logical Equivalences

Knowledge of logically equivalent statements is very useful for constructing arguments. It often happens that it is difficult to see how a conclusion follows from one form of a statement, whereas it is easy to see how it follows from a logically equivalent form of the statement. A number of logical equivalences are summarized in Theorem 2.1.1 for future reference.

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

p.35

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$

THE CONDITIONAL Statement
AND RELATED CONDITIONALS

A Conditional statement is one of the form

If p, then q.

p is called the Hypothesis and q is the Conclusion.

In Symbols: $p \to q$ means "If p, Then q".

"If p, then q" asserts "In every situation in which p is true, q is also true by coincidence."

"If p, then q" asserts "It never happens that p is true and q is false."

There is no assertion that the truth of p causes q to be true. It only asserts coincidence.

The Truth Table of If p, Then q:

p	q	$p \to q$	
T	T	T	
T	F	F	
F	T	T	(Not Applicable)
F	F	T	(Not Applicable)